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# Gelfand–Dikii system revisited

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**Abstract**

By making use of the algebraic structure of the problem, we give a complete description of the topology and bifurcations of the invariant level sets for the Gelfand–Dikii system.

**Résumé**

En utilisant la structure algébrique du problème, nous donnons une description complète de la topologie et des bifurcations des variétés invariantes du système de Gelfand–Dikii.

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## 1. Introduction

The Gelfand–Dikii system [6] is a canonical Hamiltonian system defined by the Hamiltonian

$$H = -q_1^4 - q_2^2 + 3q_1^2 q_2 - q_1 p_2^2 - 2p_1 p_2.$$

The change of variables:

$$q_1 = u + v, \qquad q_2 = uv \tag{1}$$

separates the variables of the corresponding Hamilton–Jacobi equation and allows us to find a second integral of motion

$$F = q_1^3 q_2 + q_1^2 p_2^2 + 2q_1 p_1 p_2 - p_2^2 q_2 - 2q_1 q_2^2 + p_1^2,$$

which proves the complete integrability of the system [1].


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The aim of this paper is to give a topological description of the real level sets $\mathcal{A}^{\mathbb{R}}_{(h,f)} = \{H(q,p) = h,\ \ F(q,p) = f\}$. According to the Liouville–Arnold theorem [1], each connected component of the generic level set $\mathcal{A}^{\mathbb{R}}_{(h,f)}$ is either a torus, or a cylinder if the flow is complete. We prove that, although the flow is not complete, the generic real level set $\mathcal{A}^{\mathbb{R}}_{(h,f)}$ consists of one or two cylinders. To prove our result we use the algebraic structure of the problem and the Comessati theory [2] as explained by Silhol [10].

We then describe the bifurcations of the level sets $\mathcal{A}^{\mathbb{R}}_{(h,f)}$ when the parameter $(h, f)$ passes through the bifurcation set. It turns out that the bifurcation set is just the discriminant locus of the polynomial $P(z) = z^5 + hz + f$. As the connected components of the set $\mathcal{A}^{\mathbb{R}}_{(h,f)}$ are non-compact, Fomenko's theory [5] is not applicable; then to study the bifurcations of Liouville cylinders, we use once again the algebraic structure of the problem.

The algebraic structure of the problem was described earlier by Dimitrov [3]. We also give an independent and shorter proof of his result by making use of a construction due to Jacobi and Mumford [8]. Finally we note that the description of the topology of the Gelfand–Dikii system announced in [3] is erroneous.

The paper is organized as follows: First, we explain the algebraic structure of the problem (Section 2). In Section 3, we study the topology of general level set $\mathcal{A}^{\mathbb{R}}_{(h,f)}$. In Section 4, we find the topological bifurcations of $\mathcal{A}^{\mathbb{R}}_{(h,f)}$ as the parameter $(h, f)$ passes through the bifurcation set $B$.

## 2. Algebraic structure

Let $\mathcal{A}^{\mathbb{C}}_{(h,f)}$ be the complex affine algebraic variety defined by

$$\mathcal{A}^{\mathbb{C}}_{(h,f)} = \{(q_1, q_2, p_1, p_2) \in \mathbb{C}^4 : H = h, F = f\}.$$

The Hamiltonian system corresponding to Gelfand–Dikii Hamiltonian reads:

$$\begin{aligned} \dot q_1 &= -2p_2, & \dot p_1 &= 4q_1^3 - 6q_1q_2 + p_2^2, \\ \dot q_2 &= -2p_1 - 2q_1p_2, & \dot p_2 &= 2q_2 - 3q_1^2. \end{aligned} \tag{2}$$

On $\mathcal{A}^{\mathbb{C}}_{(h,f)}$, according to (1), we obtain after some algebraic manipulations:

$$p_1 = \frac{u\sqrt{P(u)} - v\sqrt{P(v)}}{u - v}, \qquad p_2 = -\frac{\sqrt{P(u)} - \sqrt{P(v)}}{u - v}, \tag{3}$$

where

$$P(z) = z^5 + hz + f.$$

A straightforward computation shows that the Hamilton–Jacobi equation separates in these $(u, v)$-coordinates and moreover (2) is equivalent to the Jacobi inversion problem:

$$\frac{\mathrm{d}u}{\sqrt{P(u)}} + \frac{\mathrm{d}v}{\sqrt{P(v)}} = 0, \qquad \frac{u\,\mathrm{d}u}{\sqrt{P(u)}} + \frac{v\,\mathrm{d}v}{\sqrt{P(v)}} = 2\,\mathrm{d}t. \tag{4}$$

In a similar way, one computes that the system:

$$\dot q_1 = 2q_1p_2 + 2p_1, \qquad \dot q_2 = 2p_2q_1^2 + 2q_1p_1 - 2p_2q_2,$$
$$\dot p_1 = -3q_1^2q_2 - 2q_1p_2^2 - 2p_1p_2 + 2q_2^2, \qquad \dot p_2 = -q_1^3 + p_2^2 + 4q_1q_2$$

associated with the second integral $F$ is equivalent to:

$$\frac{\mathrm{d}u}{\sqrt{P(u)}} + \frac{\mathrm{d}v}{\sqrt{P(v)}} = 2\,\mathrm{d}t, \qquad \frac{u\,\mathrm{d}u}{\sqrt{P(u)}} + \frac{v\,\mathrm{d}v}{\sqrt{P(v)}} = 0. \tag{5}$$

Let $\Gamma$ be the hyperelliptic curve defined by

$$\{w^2 = P(z)\}$$

and let $(w_1, w_2)$ be a canonical basis of holomorphic differentials on $\Gamma$ :

$$w_1 = \frac{a_1z + b_1}{\sqrt{P(z)}}\,\mathrm{d}z, \qquad w_2 = \frac{a_2z + b_2}{\sqrt{P(z)}}\,\mathrm{d}z.$$

The Abel–Jacobi map is defined by

$$\begin{aligned} \zeta : \Gamma^{(2)} &\longrightarrow Jac(\Gamma) \\ P_1 + P_2 &\mapsto \left(\int_{P_0}^{P_1} w_1 + \int_{P_0}^{P_2} w_1;\ \int_{P_0}^{P_1} w_2 + \int_{P_0}^{P_2} w_2\right), \end{aligned}$$

where $Jac(\Gamma)$ is the Jacobi variety of $\Gamma$, $P_0$ is a fixed base point on $\Gamma$, and $\Gamma^{(2)}$ the symmetric product of $\Gamma$.

Solving the Jacobi inversion problem, we may express $u+v$ and $uv$ in terms of genus-two hyperelliptic theta functions associated with $\Gamma$ [3,4].

**Theorem 2.1** (see [3]).
(1) *If $(f/4)^4 + (h/5)^5 \neq 0$ then the affine algebraic variety $\mathcal{A}^{\mathbb{C}}_{(h,f)}$ is a smooth complex manifold isomorphic to $Jac(\Gamma) \setminus D_\infty$ where $D_\infty$ is a genus-two hyperelliptic curve.*
(2) *The Hamiltonian flows defined by $H$ and $F$ on $\mathcal{A}^{\mathbb{C}}_{(h,f)}$ extend biholomorphically to flows on $Jac(\Gamma)$ which are straight-line motions.*

*Proof.* Consider the polynomial $f(z)$ defined by

$$f(z) = z^5 + hz + f$$

and $U$, $V$, $W$ the Jacobi polynomials [8] associated with $\Gamma$:

$$U(z) = (z-u)(z-v), \qquad V(z) = \frac{\sqrt{f(u)}(z-v) - \sqrt{f(v)}(z-u)}{u-v},$$

$$W(z) = \frac{f(z) - V(z)^2}{U(z)}.$$

After some calculations we obtain:

$$U(z) = z^2 - q_1 z + q_2, \qquad V(z) = -p_2 z + (q_1 p_2 + p_1),$$
$$W(z) = z^3 + q_1 z^2 + (q_1^2 - q_2)z + (q_1^3 - 2q_1 q_2 - p_2^2).$$

Obviously,

$$\begin{aligned} \mathcal{A}^{\mathbb{C}}_{(h,f)} &= \{(q_1, q_2, p_1, p_2) \in \mathbb{C}^4 : H = h;\ F = f\} \\ &= \{(U, V, W) \in \mathbb{C}^7 : f(z) - V(z)^2 = U(z)W(z)\}. \end{aligned}$$

According to [8, Theorem 1.3, p.3.21], the latter is a smooth manifold if $f(z)$ has no double zero. Since, for $(f/4)^4 + (h/5)^5 \neq 0$, $f(z)$ has no double zero, then $\mathcal{A}^{\mathbb{C}}_{(h,f)}$ is smooth. Further $\mathcal{A}^{\mathbb{C}}_{(h,f)} = Jac(\Gamma) \setminus D_\infty$ [8, Theorem 10.1, p.3.157], $D_\infty$ is a translation of the divisor $\Theta$ and by Riemann's theorem:

$$D_\infty = \Theta + k = \zeta(\Gamma) + \zeta(P_\infty).$$

We shall suppose that $P_0 = P_\infty$, i.e. $\zeta(P_\infty) = 0$.

We notice that $D_\infty$ is the image by the Abel–Jacobi map of the set $F \subset \Gamma^{(2)}$ such that

$$F = \{P_\infty + P : P \in \Gamma\} \cup \{P + \tau(P) : P \in \Gamma\}.$$

$\tau$ is the hyperelliptic involution on $\Gamma$ given by

$$\tau(z, w) = (z, -w).$$

It follows that

$$Jac(\Gamma) \setminus \zeta(\Gamma) \cong \Gamma^{(2)} \setminus F. \tag{6}$$

At last, condition (2) follows from Eqs.(4) and (5). The theorem is proved. □

## 3. Topological analysis

In this section we consider (2) as a system of real differential equations and we give the topological type of the real variety

$$\mathcal{A}^{\mathbb{R}}_{(h,f)} = \{(q_1, q_2, p_1, p_2) \in \mathbb{R}^4;\ H = h,\ F = f\} \quad \text{for } (h, f) \in \mathbb{R}^2 \setminus B,$$

where

$$B = \{(h, f) \in \mathbb{R}^2 : (h/5)^5 + (f/4)^4 = 0\}.$$

**Definition 3.1.** Let $M$ be a complex algebraic variety ($M$ can be an algebraic curve). A real structure on $M$ is defined by an anti-holomorphic involution $S$. We denote this structure by $(M, S)$. The fixed points of $M$ under the action of $S$ are the real part of $M$ which will be denoted $M(\mathbb{R})$.

**Definition 3.2.** Let $(M, S)$ and $(M', S')$ be two real structures. We will say that these real structures are real isomorphic if there exists an isomorphism $\varphi : M \longrightarrow M'$ such that $S = \varphi^{-1} \circ S' \circ \varphi$.

Notice that if $(M, S)$ and $(M', S')$ are real isomorphic then $M(\mathbb{R})$ and $M'(\mathbb{R})$ are isomorphic.

We will use the following theorem due to Comessatti.

**Theorem 3.1.** *Let $C$ be an algebraic curve of genus $g$.*

(a) *If the real part of $C$, $C(\mathbb{R})$, has $r$ connected components, then the real part $Jac(C)(\mathbb{R})$ of the Jacobi variety of $C$, $Jac(C)$, is such that*

$$Jac(C))(\mathbb{R}) \cong (\mathbb{R}/\mathbb{Z})^g \times (\mathbb{Z}/2)^{r-1}.$$

(b) *If $C(\mathbb{R}) = \emptyset$ and*

(1) *if $g$ is even then*

$$Jac(C)(\mathbb{R}) \cong (\mathbb{R}/\mathbb{Z})^g,$$

(2) *if $g$ is odd then*

$$Jac(C)(\mathbb{R}) \cong (\mathbb{R}/\mathbb{Z})^g \times (\mathbb{Z}/2).$$

*In other words, the number of connected components of $Jac(C)(\mathbb{R})$ is 1 or 2 or $2^{r-1}$.*

*Proof.* See [2,10]. □

Let $S$ be the involution on $\mathcal{A}^{\mathbb{C}}_{(h,f)}$ defined by

$$\begin{array}{rcl} S : \mathcal{A}^{\mathbb{C}}_{(h,f)} & \longrightarrow & \mathcal{A}^{\mathbb{C}}_{(h,f)} \\ (q_1, q_2, p_1, p_2) & \mapsto & (\bar{q_1}, \bar{q_2}, \bar{p_1}, \bar{p_2}). \end{array} \tag{7}$$

$S$ coincides with the complex conjugation on $\mathbb{C}^4$ and gives a real structure on $\mathcal{A}^{\mathbb{C}}_{(h,f)}$ :

$$\begin{aligned} \mathcal{A}^{\mathbb{C}}_{(h,f)}(\mathbb{R}) &= \mathcal{A}^{\mathbb{R}}_{(h,f)} \\ &= \{(q_1, q_2, p_1, p_2) \in \mathcal{A}^{\mathbb{C}}_{(h,f)} : S(q_1, q_2, p_1, p_2) = (q_1, q_2, p_1, p_2)\}. \end{aligned} \tag{8}$$

$S$ induces an involution $\sigma$ on $Jac(\Gamma)$ which gives a real structure on it such that: for $(q_1, q_2, p_1, p_2) \in \mathcal{A}^{\mathbb{C}}_{(h,f)}$ parameterized by $u$, $v$ as in (1) and (3)

$$P = u + v \in \Gamma^{(2)}, \qquad \sigma(\zeta(P)) = \zeta(S(P)). \tag{9}$$

**Proposition 3.1.** *Denote by $\overline{\mathcal{A}^{\mathbb{C}}_{(h,f)}}$ the compacified of $\mathcal{A}^{\mathbb{C}}_{(h,f)}$. Then:*

(1) *$(\overline{\mathcal{A}^{\mathbb{C}}_{(h,f)}}, S)$ and $(Jac(\Gamma), \sigma)$ are real isomorphic.*

(2) (a) *If $(h/5)^5 + (f/4)^4 > 0$ then*

$$(Jac(\Gamma))(\mathbb{R}) \cong (\mathbb{R}/\mathbb{Z})^2,$$

*i.e. one real torus.*

(b) *If* $(h/5)^5 + (f/4)^4 < 0$ *then*

$$(Jac(\Gamma))(\mathbb{R}) \cong (\mathbb{R}/\mathbb{Z})^2 \times (\mathbb{Z}/2),$$

*i.e. two real tori.*

*Proof.* According to Theorem 2.1, $\overline{\mathcal{A}^{\mathbb{C}}_{(h,f)}}$ is a complex abelian surface isomorphic to $Jac(\Gamma)$ and it has real points for all values of $(h, f)$; then $(\overline{\mathcal{A}^{\mathbb{C}}_{(h,f)}}, S)$ is real isomorphic to $(Jac(\Gamma), \sigma)$ [9, Theorem 3.1].

The second part of this theorem results from Theorem 3.1. Indeed:

(a) If $(h/5)^5 + (f/4)^4 > 0$, $P(z)$ has one real root, $z_0$, and therefore, the real part of the curve $\Gamma$, $\Gamma(\mathbb{R})$, has one connected component on $[z_0; \infty)$. Then one obtains

$$(Jac(\Gamma))(\mathbb{R}) \cong (\mathbb{R}/\mathbb{Z})^2 \times (\mathbb{Z}/2)^0.$$

(b) If $(h/5)^5 + (f/4)^4 < 0$ then $P(z)$ has three simple roots: $z_1 < z_2 < z_0$, and so $\Gamma(\mathbb{R})$ has two connected components on $[z_1, z_2]$ and $[z_0, \infty)$; therefore,

$$(Jac(\Gamma))(\mathbb{R}) \cong (\mathbb{R}/\mathbb{Z})^2 \times (\mathbb{Z}/2)^{2-1}.$$

The proposition is proved. □

**Corollary 3.1.** *Denote by* $\zeta(\Gamma)(\mathbb{R})$ *the real part of* $\zeta(\Gamma)$.

(a) *If* $(h/5)^5 + (f/4)^4 > 0$ *then*

$$\mathcal{A}^{\mathbb{R}}_{(h,f)} \cong (\mathbb{R}/\mathbb{Z})^2 \setminus \zeta(\Gamma)(\mathbb{R}).$$

(b) *If* $(h/5)^5 + (f/4)^4 < 0$ *then*

$$\mathcal{A}^{\mathbb{R}}_{(h,f)} \cong \{(\mathbb{R}/\mathbb{Z})^2 \times (\mathbb{Z}/2)\} \setminus \zeta(\Gamma)(\mathbb{R}).$$

*Proof.* This result is a consequence of Theorem 2.1 and Proposition 3.1(1). □

**Theorem 3.2.**

(a) *If* $disc(P(z)) > 0$ *then* $\mathcal{A}^{\mathbb{R}}_{(h,f)}$ *is a cylinder.*

(b) *If* $disc(P(z)) < 0$ *then* $\mathcal{A}^{\mathbb{R}}_{(h,f)}$ *is a union of two cylinders.*

*Proof.* If $disc(P(z)) > 0$, $\Gamma(\mathbb{R})$ has one connected component on $[z_0; \infty)$ which we denote $\Gamma_1$. Let us also denote $\Gamma_1^{(2)}$ the symmetric product of this component, i.e.

$$\Gamma_1^{(2)} = \{P_1 + P_2 \in \Gamma^{(2)} : P_j = (u_j, w_j),\ \ u_j \in [z_0; \infty),\ \ w_j^2 = P(u_j),\ \ j = 1, 2\}$$

and $\Gamma_0^{(2)}$, the subset of $\Gamma^{(2)}$ defined by

$$\Gamma_0^{(2)} = \{P + S(P) \in \Gamma^{(2)} : P = (u, w),\ \ S(P) = (\bar{u}, \bar{w}),\ \ u \in \mathbb{C} \setminus \mathbb{R}\}.$$

According to (1) and (3), we have

$$P_i + P_j \in \Gamma_1^{(2)} \cup \Gamma_0^{(2)} \iff S(P_i + P_j) = P_i + P_j,$$

and according to (9)

$$\sigma(\zeta(P_i+P_j)) = \zeta(S(P_i+P_j)) = \zeta(P_i+P_j).$$

Then by the Abel–Jacobi map

$$Jac(\Gamma)(\mathbb{R}) \cong \Gamma_1^{(2)} \cup \Gamma_0^{(2)}. \tag{10}$$

Denote by:

$$\Gamma_{1,\infty} = \{P+P_\alpha \in \Gamma^{(2)} : P \in \Gamma_1 \text{ and } P_\alpha = \tau(P) \text{ or } P_\alpha = P_\infty\} \subset \Gamma^{(2)},$$

$$D_\infty(\mathbb{R}) = \zeta(\Gamma_{1,\infty}) = \zeta(\Gamma_1).$$

Then

$$Jac(\Gamma)(\mathbb{R}) \cap D_\infty(\mathbb{R}) = \zeta(\Gamma_1).$$

The embedding

$$\zeta : \Gamma \hookrightarrow Jac(\Gamma) \quad P \mapsto \zeta(P)$$

induces a homomorphism

$$\varphi : H_1(\Gamma, \mathbb{Z}) \longrightarrow H_1(Jac(\Gamma), \mathbb{Z}),$$

which is in fact an isomorphism.

As $\Gamma_1$ represents a non-zero homology cycle on $\Gamma$, $\zeta(\Gamma_1)$ represents a non-zero homology cycle on $Jac(\Gamma)$ too.

We conclude that if we remove this circle from the real torus $Jac(\Gamma)(\mathbb{R})$, we obtain a cylinder.

If $disc(P(z) < 0$, then the real part of the curve $\Gamma$, $\Gamma(\mathbb{R})$, has two connected components which we denote by $\Gamma_1$ and $\Gamma_2$, respectively, on $[z_0; \infty)$ and $[z_1; z_2]$.

According to Abel–Jacobi map, formulas (1),(3), and the remark mentioned above we have

$$Jac(\Gamma)(\mathbb{R}) \cong \underbrace{(\Gamma_1 \times \Gamma_2)}_{T_1} + \underbrace{(\Gamma_1^{(2)} \cup \Gamma_2^{(2)} \cup \Gamma_0^{(2)})}_{T_2}, \tag{11}$$

where

$$\Gamma_2^{(2)} = \{P_1+P_2 \in \Gamma^{(2)} : P_j = (u_j, w_j),\ \ u_j \in [z_1; z_2],\ \ w_j^2 = P(u_j),\ \ j = 1,2\}.$$

$T_1$ and $T_2$ have no common point because of: for $z \in ]z_2, z_0[$, if $(u+v) \in T_1$ then $(z-u)(z-v) < 0$ and if $(u+v) \in T_2$, $(z-u)(z-v) > 0$.

Define:

$$\Gamma_{1,\infty} = \{P+P_\alpha \in \Gamma^{(2)} : P \in \Gamma_1 \text{ and } P_\alpha = \tau(P) \text{ or } P_\alpha = P_\infty\} \subset \Gamma_1^{(2)},$$

$$\Gamma_{2,\infty} = \{P+P_\infty \in \Gamma^{(2)} : P \in \Gamma_2\} \subset \Gamma_1 \times \Gamma_2.$$

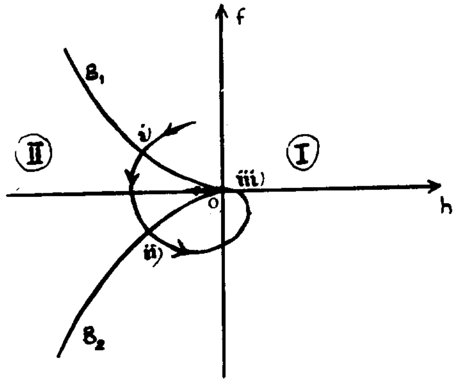


Fig. 1.

$D_\infty(\mathbb{R})$ has two connected components without common point which are $\zeta(\Gamma_{1,\infty})$ and $\zeta(\Gamma_{2,\infty})$:

$$Jac(\Gamma)(\mathbb{R}) \cap D_\infty(\mathbb{R}) = \zeta(\Gamma_{1,\infty}) \cup \zeta(\Gamma_{2,\infty})$$

and

$$T_1 \cap D_\infty(\mathbb{R}) = \zeta(\Gamma_{2,\infty}) = \zeta(\Gamma_2), \qquad T_2 \cap D_\infty(\mathbb{R}) = \zeta(\Gamma_{1,\infty}) = \zeta(\Gamma_1).$$

As before, $\zeta(\Gamma_1)$ and $\zeta(\Gamma_2)$ are two circles on $Jac(\Gamma)$ not homologous to zero.

We conclude that

$$Jac(\Gamma)(\mathbb{R}) \setminus \zeta(\Gamma)(\mathbb{R}) \cong \{T_1 \setminus S^1\} \cup \{T_2 \setminus S^1\}$$

is a union of two cylinders. The theorem is proved. □

## 4. Topological bifurcations of the varieties $\mathcal{A}^{\mathbb{R}}_{(h,f)}$

In this section we describe the topology of singular level sets $\mathcal{A}^{\mathbb{R}}_{(h,f)}$, when $(h, f) \in B$. We recall that $B$ is defined by

$$B = \{(h, f) \in \mathbb{R}^2 : (h/5)^5 + (f/4)^4 = 0\}.$$

We denote by $B_1$ and $B_2$ the connected components of the set $B$ corresponding, respectively, to $f > 0$ and $f < 0$ (Fig. 1).

If $(h_0, f_0) \in B_1$ then $P(z)$ has a double real root $z_0 = z_2$ and a simple root $z_1 < z_0$; and for $(h_0, f_0) = (0, 0)$, 0 is a root of multiplicity 5.

Let $L_1$ and $L_0$ denote the sets:

$$L_0 = \{0 + P : \ P \in \Gamma_1 \setminus P_\infty \ \ 0 = (0, 0) \ \} \subset \Gamma^{(2)},$$

$$L_1 = \{P_0 + P : P \in \Gamma_1 \setminus P_\infty \ \text{and} \ P_0 = (z_0; 0)\} \subset \Gamma^{(2)},$$

where $\Gamma_1$ is the only connected component of the real part of $\Gamma$ when $(h_0, f_0) \in B_1 \cup (0, 0)$ (see Figs. 2b.b and 4b.b).

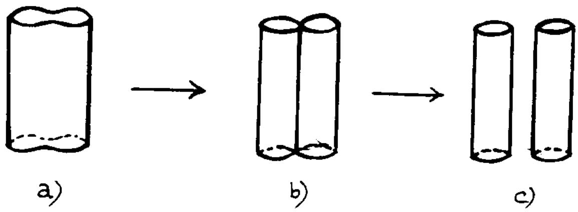


Fig. 2a.

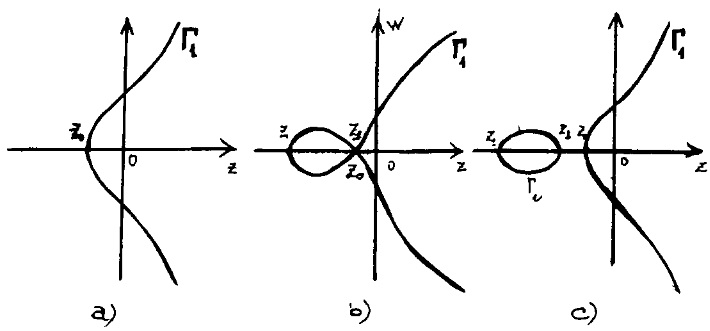


Fig. 2b.

In the same way, if $(h_0, f_0) \in B_2$, $P(z)$ has a double real root $z_1 = z_2$ and a simple root $z_0 > z_1$. $L_2$ will represent the set

$$L_2 = \{P_1 + P : P \in \Gamma_1 \setminus P_\infty \text{ and } P_1 = (z_1; 0)\} \subset \Gamma^{(2)}.$$

$\Gamma_1$ is also the connected component of the curve $\Gamma$ on $[z_0; \infty)$ (See Fig. 3b.b).

**Theorem 4.1.** *Let $(h_0, f_0) \in B$ and $\mathcal{A}^{\mathbb{R}}_{(h_0; f_0)}$ be the corresponding singular level set.*

(a) *If $(h_0, f_0) \in B_1 \cup \{(0, 0)\}$ then $\mathcal{A}^{\mathbb{R}}_{(h_0; f_0)}$ is a smooth real manifold, except in the points $(q_1, q_2, p_1, p_2)$ parameterized by $(u_0, v) \in L_1$ or $L_0$.*
   - *If $(h_0, f_0) \in B_1$, $\mathcal{A}^{\mathbb{R}}_{(h_0; f_0)}$ is homeomorphic to $(S^1 \vee S^1) \times \mathbb{R}^1$ where $S^1 \vee S^1$ is a union of two circles with one common point* (Fig. 2a.b).
   - *If $(h_0, f_0) = (0, 0)$, $\mathcal{A}^{\mathbb{R}}_{(h_0; f_0)}$ is a cylinder with singularities along the line $\zeta(L_0)$* (Fig. 4a.b).

(b) *If $(h_0, f_0) \in B_2$ then $\mathcal{A}^{\mathbb{R}}_{(h_0; f_0)}$ is a union of a cylinder and a line* (Fig. 3a.b).

*Proof.* According to Theorem 2.1, in all the points $(q_1, q_2, p_1, p_2) \in \mathcal{A}^{\mathbb{R}}_{(h_0; f_0)}$ which are not parameterized by $(u_0, v_0) \in L_k$, $k = 0, 1, 2$, $\mathcal{A}^{\mathbb{R}}_{(h_0; f_0)}$ is smooth. Now, we show that $L_0$, $L_1$ and $L_2$ parameterize the singular points of $\mathcal{A}^{\mathbb{R}}_{(h_0, f_0)}$. According to (1) the Hamiltonian $H$ becomes

$$H(u, v, p_u, p_v) = -u^4 - u^3 v - u^2 v^2 - u v^3 - v^4 + \frac{p_u^2}{(u - v)} - \frac{p_v^2}{(u - v)},$$

where $p_u$ and $p_v$ are the conjugate coordinates of $u, v$.

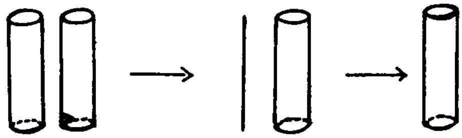


Fig. 3a.

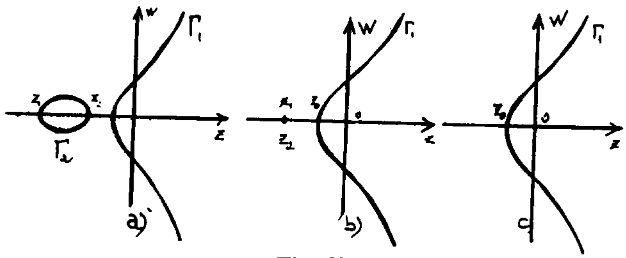

Fig. 3b.

We also have

$$(u - v)H = -u^5 + v^5 + p_u^2 - p_v^2,$$

and hence

$$F = -uH - u^5 + p_u^2.$$

The calculation of the gradient of $F$ gives

$$grad(F) = \begin{cases} \dfrac{\partial F}{\partial u} = -H - 5u^4 - u\dfrac{\partial H}{\partial u}, \\[2ex] \dfrac{\partial F}{\partial v} = -u\dfrac{\partial H}{\partial v}, \\[2ex] \dfrac{\partial F}{\partial p_u} = 2p_u - u\dfrac{\partial H}{\partial p_u}, \\[2ex] \dfrac{\partial F}{\partial p_v} = -u\dfrac{\partial H}{\partial p_v}. \end{cases} \tag{12}$$

It is obvious that if:

$$P'(u) = h + 5u^4 = 0, \qquad p_u = \tfrac{1}{2}(u - v)\dot{u} = 0, \tag{13}$$

then

$$grad(F) = -u\, grad(H).$$

So, as in the points of $\mathcal{A}^{\mathbb{R}}_{(h_0;f_0)}$ parameterized by $(u_0, v) \in L_k$, $k = 0, 1, 2$, we have

$$\begin{aligned} &P(u_0) = u_0^5 + hu_0 + f = 0, \qquad P'(u_0) = h + 5u_0^4 = 0, \\ &p_u = \tfrac{1}{2}(u_0 - v)\dot{u}_0 = 0, \end{aligned} \tag{14}$$

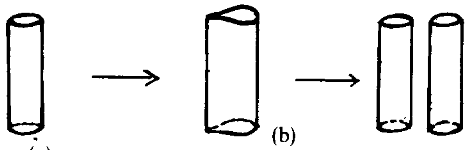

Fig. 4a.

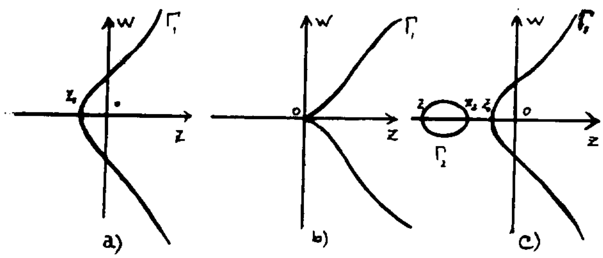

Fig. 4b.

then

$$grad(F) = -u_0\, grad(H).$$

It follows that, at these points, $\mathcal{A}^{\mathbb{R}}_{(h_0;f_0)}$ is not smooth.

Let us consider the map $\mu$ defined by

$$\begin{aligned} \mu : \Gamma^{(2)} \setminus F &\longrightarrow \mathcal{A}^{\mathbb{R}}_{(h_0;f_0)} \\ u + v &\mapsto (q_1, q_2, p_1, p_2), \end{aligned}$$

where $q_1, q_2, p_1$ and $p_2$ are as in (1) and (3).

If $(h_0, f_0) \in B_1 \cup \{(0,0)\}$,

$$\mathcal{A}^{\mathbb{R}}_{(h_0;f_0)} \cong \{\Gamma_1^{(2)} \cup \Gamma_0^{(2)}\} \setminus (\Gamma_{1,\infty}).$$

For $(h_0, f_0) \in B_2$ according to the map $\mu$,

$$\mathcal{A}^{\mathbb{R}}_{(h_0,f_0)} \cong \underbrace{\{P_1\} \times \{\Gamma_1 \setminus P_\infty\}}_{(L_2)} + \underbrace{\{\Gamma_1^{(2)} \cup \Gamma_0^{(2)}\} \setminus (\Gamma_{1,\infty})}_{(C)}.$$

The theorem is proved. □

**Corollary 4.1.** *All bifurcations of connected components of the invariant manifold $\mathcal{A}^{\mathbb{R}}_{(h,f)}$ are described in Fig. 1.*

*Proof.* Bifurcation (i): The cylinder obtained in domain (I) collapses along the axial line ($\zeta(L_1)$, Theorem 4.1) before splitting into two cylinders. This bifurcation is described in Fig. 2a

Bifurcation (ii): It is the "inverse" bifurcation of the following bifurcation: one of two cylinders contracts to this axial line ($\zeta(L_1)$, Theorem 4.1) and "vanishes". This bifurcation is described in Fig. 3a

Bifurcation (iii): The cylinder collapses along the line $\zeta(L_0)$ before splitting into two cylinders. This bifurcation is represented by Fig. 4a.

The corollary is proved. □

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